Two Problems from IIT-JEE

IIT-JEE (Joint Entrance Examination for admission to Indian Institutes of Technology) is the most prestigious examination at 10+2 level in India. The questions are supposed to be tough and require ingenuity on the part of the student. There is a booming ecosystem of books, coaching classes and correspondence classes around IIT-JEE. But I find this ecosystem to be quite detrimental in improving the abilities of the student.

Today I will discuss two problems from the mathematics paper which I think are worth discussing here. The reason behind choosing these two problems is the fact that both of them are quite old (one from IIT-JEE 1981 and another from IIT-JEE 2001), but inspite of them being solved I have not yet found a proper solution to them in any of the books especially being published for IIT-JEE. This also shows the flaw in the ecosystem developed around IIT-JEE which focuses only on problem solving without providing any conceptual framework.

To begin with, let me start with an easy (seemingly) problem from Maths paper of IIT JEE 1981.
Problem 1: Let $f(x)$ be a function defined for all $x \in \mathbb{R}$ such that $f(x+y) = f(x)\cdot f(y)$ for all $x, y \in \mathbb{R}$. Let $f'(0) = 3$ and $f(5) = 2$. Find the value of $f'(5)$.

It is easy to observe that $f'(x)$ exists for all $x \in \mathbb{R}$. For, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x)f(h)-f(x)f(0)}{h}\\ &= \lim_{h\to 0} f(x)\cdot\frac{f(h)-f(0)}{h} = f(x)f'(0) = 3f(x)\end{aligned}$$

and therefore $f'(5) = 3f(5) = 3\cdot 2 = 6$. This is exactly the solution provided in all the books which I have seen in the Indian market.

If we just dig slightly deeper we will find something really strange. A simple look at the functional equation

$$f(x+y) = f(x)f(y)$$

gives us a hint that $f(x)$ is an exponential function. To make the argument rigorous we observe that

$$\frac{d}{dx}\left(\frac{f(x)}{e^{3x}}\right) = \frac{e^{3x}f'(x) - 3e^{3x}f(x)}{e^{6x}} = 0$$

as $f'(x) = 3f(x)$. It follows that $f(x) = ke^{3x}$ where k is some constant. Now $f(0) = k$ and since $f(0) = f(0+0) = f(0)f(0)$ it follows that $k = k\cdot k = k^2$. Thus either $k = 0$ or

$k = 1$. If $k = 0$ then $f(x) = 0$ for all $x \in \mathbb{R}$ and therefore $f'(x) = 0$ for all $x \in \mathbb{R}$, which is not the case here. We thus have $k = 1$ and $f(x) = e^{3x}$.

Thus $f(5) = e^{15} > e > 2$ and there is no way that $f(5) = 2$. The function mentioned in the problem does not exist. But I don't think even the examiner had this thought in his mind. I had encountered this problem way back in 1997 from some book for IIT-JEE and had told the above solution to many people but no one was convinced that the books or the JEE question were wrong. There are scores of books on calculus for IIT-JEE which I have seen, but none of them have expressed the point which I have made about this question.

This does not sound good as the current students at 10+2 level are being programmed to act like mechanical robots and crack JEE. Especially in the field of calculus most of the students are conceptually bankrupt and for no fault of theirs. The books available for competitions simply try to teach them algebraical manipulations along with magical formulas of calculus and expect the student to be a master in the art of solving problems. Next they are provided with lots and lots of problems (mostly solved and few unsolved something like 70/30 ratio) so that they can tackle any problem which is similar to the ones provided. An average calculus book is therefore spanning around 1000 to 1200 pages. Coaching classes are even worse as they provide even larger set of problems and tests and the student acts more like a forced laborer trying to keep all the problems and their solutions in his head.

The second problem I have chosen is from IIT-JEE 2001 and I happened to come across it a few days ago when I was teaching calculus to my cousin. (After clearing JEE in 1998, I didn't have the chance to look at JEE problems with that much curiosity)
Problem 2: Let $f(x)$ be a non-negative continuous function defined for $x \geq 0$ and $F(x) = \int_0^x f(t)dt$. Also let there be a constant $c > 0$ such that $f(x) \leq cF(x)$ for all $x \geq 0$. Prove that $f(x) = 0$ for all $x \geq 0$.

All the solutions which I have seen till now proceed in very crazy fashion. First they assume that $f(x) = 0$ for all $x \geq 0$ and then there is nothing to prove. If that's not the case they assume (by magic and not logic) that $f(x) > 0$ for all $x \geq 0$. This is real crazy stuff. **If its not the case that $f(x) = 0$ for all $x \geq 0$ then it means that there is some number say $x_0 > 0$ such that $f(x_0) > 0$.** I doubt the authors of these books really understand what they are writing.

Moreover I think that trying to prove that $F(x) = 0$ for all $x \geq 0$ is easier than directly focusing on $f(x)$ as the anti-derivative $F(x)$ is a smoother function with many nice properties like:
1. $F(x)$ is differentiable for $x \geq 0$ (and therefore continuous too).
2. $F(x)$ is increasing (not in the strict sense) function of x for $x \geq 0$. (As $F'(x) = f(x) \geq 0$.)

Clearly $F(0) = 0$. Since $F(x)$ is increasing therefore $F(x) \geq 0$ for all $x \geq 0$. Let us assume that there is a number $b > 0$ such that $F(b) > 0$. Let us now examine the behavior of $F(x)$ in the interval $[0, b]$. Since $F(0) = 0$ there must be a last value of $x \in [0, b]$ such that $F(x) = 0$ (**this is a very subtle point and conveys much more than the intermediate value theorem for continuous functions**, see this post). Let this value of x be denoted by a. Clearly $0 \leq a < b$ and $F(a) = 0$ and $F(x) \neq 0$ for any value of $x \in (a, b]$. Since $F(x) \geq 0$ it follows that $F(x) > 0$ for all $x \in (a, b]$.

Let $x \in (a, b]$. Then we have

$$\begin{aligned}
&f(x) \leq cF(x)\\
&\Rightarrow F'(x) \leq cF(x)\\
&\Rightarrow \frac{F'(x)}{F(x)} \leq c\\
&\Rightarrow \int_{x}^{b} \frac{F'(t)}{F(t)}\,dt \leq \int_{x}^{b} c\,dt\\
&\Rightarrow \log F(b) - \log F(x) \leq c(b - x)
\end{aligned}$$

If we now let $x \to a+$ we see that the LHS of above inequality tends to ∞ and the RHS tends to a positive number $c(b - a)$. This contradiction proves that that there is no number $b > 0$ such that $F(b) > 0$. Thus $F(x) = 0$ for all $x \geq 0$ and consequently its derivative $f(x) = 0$ for all $x \geq 0$.

The subtle point mentioned in bold is not presented in common textbooks of calculus. What is sometimes given without proof is the following:
If $f(x)$ is continuous in closed interval $[a, b]$ and $f(a)f(b) < 0$ there is a number $c \in (a, b)$ such that $f(c) = 0$.

Without using the fact that continuous functions take their values for the first and last time in a closed interval it is difficult to solve the above problem. At least I am not aware of any solution which avoids this rarely mentioned property of continuous function. Readers are welcome to provide any other solution to this problem.

P.S.: I received another solution to the second problem in the comments which I found so beautiful as to include in the post itself. Thanks to Milind Hegde for sharing this solution which avoids some deep theorems which I have used in my solution.

As before the idea is to prove that the antiderivative $F(x) = 0$. But here we use a trick of multiplying by $e^{-cx} > 0$. Thus we have the following derivation:

$$\begin{aligned}&f(x) \leq cF(x) \text{ for all } x \geq 0\\&\Rightarrow F'(x) \leq cF(x)\\&\Rightarrow e^{-cx}F'(x) - ce^{-cx}F(x) \leq 0\\&\Rightarrow \frac{d}{dx}\{e^{-cx}F(x)\} \leq 0\\&\Rightarrow \int_0^x \frac{d}{dt}\{e^{-ct}F(t)\}\,dt \leq \int_0^x 0\,dt = 0\\&\Rightarrow e^{-cx}F(x) - e^{-c\cdot 0}F(0) \leq 0\\&\Rightarrow e^{-cx}F(x) \leq 0\\&\Rightarrow F(x) \leq 0\end{aligned}$$

and since we already know that $F(x) \geq 0$ it follows that $F(x) = 0$ and therefore $f(x) = F'(x) = 0$ for all $x \geq 0$. A generalization of this problem has been discussed on MSE and a beautiful answer is also presented on MSE.

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Labels: Mathematical Analysis